A PATHOLOGICAL EXAMPLE OF A UNIFORM QUOTIENT MAPPING BETWEEN EUCLIDEAN SPACES

OLGA MALEVA

ABSTRACT. A uniform quotient Lipschitz mapping between Euclidean spaces of dimensions n and $n-1$, which annihilates the unit ball of a hyperplane, is constructed.

1. **Introduction.** This work is inspired by the paper [BJLPS], where Lipschitz quotient mappings and uniform quotient mappings are studied. A map $f\colon X \to Y$, where X and Y are metric spaces, is called a uniform quotient if

$$B_{\Omega(r)}(f(x)) \supset f(B_r(x)) \supset B_{\omega(r)}(f(x))$$

for any $x \in X$ and $r > 0$, where $\omega(r)$, $\Omega(r)$ are functions of the radius r independent of the point x, such that $\omega(r) > 0$ for $r > 0$ and $\Omega(r) \to 0$ as $r \downarrow 0$. If the first inclusion holds, f is called uniformly continuous; if the second holds, f is called co-uniformly continuous or co-uniform. If $\omega(r) \geq cr$, $\Omega(r) \leq Cr$ for some $c, C > 0$, f is said to be a Lipschitz quotient mapping (co-Lipschitz if the first inequality holds and Lipschitz if the second inequality holds).

There is a developed theory of uniform / Lipschitz quotient mappings which are one-to-one ([BL]), but not much is known in the general case.

For example, if X, Y are Banach spaces then the Gorelik principle ([G], [JLS]) says, that one-to-one uniform quotient mapping cannot carry the unit ball in a finite codimensional subspace of X into a "small" neighborhood of an infinite codimensional subspace of Y. The proof of the Gorelik principle actually shows that a bi-uniform homeomorphism cannot map a ball in a subspace of codimension k into a small neighborhood of a subspace of codimension $k+1$. This holds regardless of whether X and Y are finite or infinite dimensional.

One may ask, if a similar principle holds for uniform quotient mappings, which are not one-to-one. It turns out, that this is not the case even for finite dimensional spaces.

Supported by the Israel Science Foundation.

As it was proved in [BJLPS], for each n there is a uniform quotient mapping from $\mathbb{R}^{2n+1}$ onto $\mathbb{R}^n$ which maps the unit ball of the hyperplane to zero. Moreover, there is a stronger example for low dimensions: A Lipschitz and co-uniform mapping from $\mathbb{R}^3$ onto $\mathbb{R}^2$ which annihilates the unit ball of a hyperplane.

In the present paper we generalize this construction to the case of arbitrary dimension. The result of the paper reads as follows:

For $n \geq 1$ there is a Lipschitz and co-uniform mapping T from $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \mathbb{R}$ onto $\mathbb{R}^{n+1}$ such that $T(B_1^{\mathbb{R}^{n+1}\oplus 0}(0)) = \{0\}$.

2. **The idea of the construction.** Before going into the technical details we briefly describe the example and the proof in an informal way. The space $\mathbb{R}^{n+2}$ is decomposed into the direct sum $\mathbb{R}^{n+1} \oplus \mathbb{R} = \{(x, a) \mid x \in \mathbb{R}^{n+1}, a \in \mathbb{R}\}$, and the mapping is of the form $T(x,a) = \varphi_a(\|x\|) \cdot U_{\psi_a(\|x\|)}x$, where $U_{(\cdot)}$ is a family of orthogonal operators acting on $\mathbb{R}^{n+1}$. This family together with the functions $\varphi_a(\|x\|)$ and $\psi_a(\|x\|)$ are chosen in such a way that the mapping T is clearly Lipschitz.

The main part of the proof deals with the co-uniformity of T, namely we check the inclusion $TB_r(x,a) \supset B_{\omega(r)}(T(x,a))$ for a fixed radius $r > 0$. It turns out that if a or $\|x\|$ is large enough, more exactly if $\|x\| > 1 + \alpha_1 r^n$ or if $|a| > \alpha_2 r$ for suitably chosen constants α_1 and α_2, then for a fixed and y close to $f_a(x) = T(x,a)$ in $\mathbb{R}^n$, the gradient of $f_a^{-1}(y)$ is uniformly bounded in norm by a certain constant c, depending on r. So $TB_r(x,a) \supset T(B_r(x),a) \supset B_{r/c}(T(x,a))$.

The other case is: $\|x\|$ is less than 1 (or not much greater than 1) and $|a| \leq \alpha_2 r$. In this case the inclusion $TB_r(x,a) \supset B_{\omega(r)}$ is of different nature. If x remains fixed and a runs over $[\,0, \alpha_2 r\,]$ (so the point (x,a) does not leave the ball of radius r), the point $T(x,a)$ "draws" a curve which is "dense" in the ball $B_{\|x\|c(r)}(0)$ in the sense that its small neighborhood contains $B_{\|x\|c(r)}(0) \supset B_{\omega(r)}(T(x,a))$. This small neighborhood is contained, say, in the image of $B_{r/2}(x) \times [\,0, \alpha_2 r\,] \subset B_r(x,a)$, so the inclusion follows. This remarkable Lipschitz curve $T(x, [\,0, \alpha_2 r\,])$ looks like a spiral of infinitely many turns around 0, when $x \in \mathbb{R}^2$ (see Fig. 1 below). In higher dimensions the curve is some spatial analogue of such a spiral.

In this part we use a special lemma, which allow us to approximate a fixed finite sequence of angles by residues of $\frac{2\pi}{\gamma}$, $\frac{2\pi}{\gamma^2}$, $\ldots$,$\frac{2\pi}{\gamma^n}$ modulo 2π.

The question, whether there exists a Lipschitz quotient mapping from $\mathbb{R}^n$ onto $\mathbb{R}^m$ which annihilates an object of dimension greater than $n - m$, remains open.

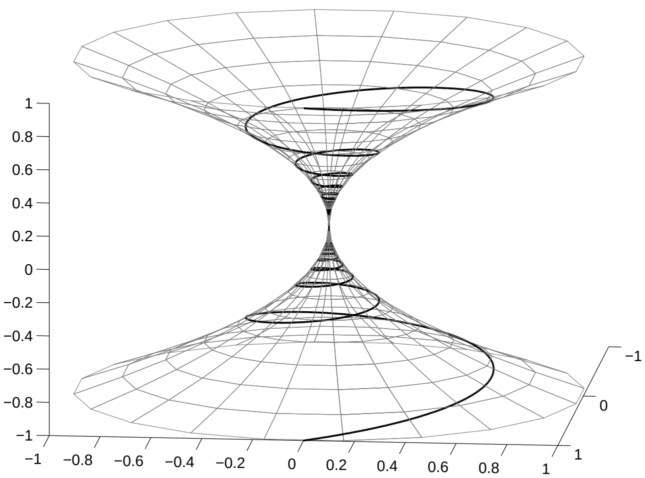

FIGURE 1. The image $T((0,1),a)$, $-1 \leq a \leq 1$ is the projection of the bolded curve onto the bottom plane

This work will form part of a Ph.D. thesis written under the supervision of Professor Gideon Schechtman. The author thanks Professor Gideon Schechtman for suggesting the problem and for useful discussions.

3. **The construction.**

Theorem. For $n \geq 1$ there is a Lipschitz mapping T from $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \mathbb{R}$ onto $\mathbb{R}^{n+1}$ such that T is a co-uniform quotient mapping and $T(B_1^{\mathbb{R}^{n+1}\oplus 0}(0)) = \{0\}$.

Proof. Let x_k be the kth coordinate vector of the space $\mathbb{R}^{n+1}$, and Ox_kx_{k+1} denote the coordinate plane spanned by x_k, x_{k+1}. We interpret $\mathbb{R}^k$ as the subspace of $\mathbb{R}^{n+1}$ spanned by $x_1, \ldots, x_k$. Denote by π_k the standard orthogonal projection $\mathbb{R}^{n+1} \to \mathbb{R}^k$. Let S_r^k denote a sphere in $\mathbb{R}^{k+1}$ of radius r, centered at zero. By $R^{\alpha}_{Ox_kx_{k+1}}$ we mean the orthogonal transformation of the space, which acts as planar rotation by α in the kth and $(k+1)$th coordinates, leaving the rest of the coordinates unchanged. Note that

$$(1) \quad \text{if } \|v\| = \|w\| \text{ and } v - w \in Ox_kx_{k+1}, \quad \text{then } w = R^{\alpha}_{Ox_kx_{k+1}} v \text{ for some } \alpha \in [0, 2\pi].$$

We define the orthogonal operator $U^{[k+1]}_{\alpha_1,\dots,\alpha_k}$ inductively by

$$U^{[2]}_{\alpha} = R^{\alpha}_{Ox_1x_2},$$
$$U^{[k+1]}_{\alpha_1,\dots,\alpha_k} = (U^{[k]}_{\alpha_2,\dots,\alpha_k})^{-1} R^{\alpha_1}_{Ox_kx_{k+1}} U^{[k]}_{\alpha_2,\dots,\alpha_k}.$$

For x fixed and α_j running over $[0,2\pi]$ independently, $U^{[n+1]}_{\alpha_1,\dots,\alpha_n}(x)$ runs over the whole sphere in $\mathbb{R}^{n+1}$ of radius $\|x\|$, centered at the origin.

To show this, let us note first that $\{U^{[2]}_{\alpha}(x) \mid \alpha \in [0,2\pi]\} = S^1_{\|x\|}$ for $x \in \mathbb{R}^2$. Assume that $U^{[k]}_{\alpha_1,\dots,\alpha_{k-1}}(x)$ runs over the whole sphere $S^{k-1}_{\|x\|}$ for fixed $x \in \mathbb{R}^k$. Now fix $x \in \mathbb{R}^{k+1}$ and take arbitrary $y \in S^k_{\|x\|}$. Since $\pi_k(x-y) \in \mathbb{R}^k$, there exist $\alpha_2,\dots,\alpha_k$ such that $U^{[k]}_{\alpha_2,\dots,\alpha_k}\pi_k(x-y) = \pi_k U^{[k]}_{\alpha_2,\dots,\alpha_k}(x-y) = \|\pi_k(x-y)\|x_k$. Then $U^{[k]}_{\alpha_2,\dots,\alpha_k}(x-y)$ lies in Ox_kx_{k+1}. By (1), there exists α_1 such that

$$U^{[k]}_{\alpha_2,\dots,\alpha_k} y = R^{\alpha_1}_{Ox_kx_{k+1}} U^{[k]}_{\alpha_2,\dots,\alpha_k} x.$$

By definition this means that $U^{[k+1]}_{\alpha_1,\dots,\alpha_k} x = y$.

For $u \in \mathbb{R}$, let $d_u \colon \mathbb{R}_+ \to [0,1]$ be the continuous function such that $d_u(t) = \min(|u|,1)$ for $t \le 1$, $d_u(t) = 1$ for $t \ge 2$, $d_u(t)$ is linear for $1 \le t \le 2$.

Define $T \colon \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ by

$$T(x,a) = d^2_{a^n}(\|x\|) U^{[n+1]}_{2\pi/d_a(\|x\|), 2\pi/d_{a^2}(\|x\|), \dots, 2\pi/d_{a^n}(\|x\|)} x.$$

Note that for $n = 1$ this reduces to the construction in [BJLPS].

Let us check that T is a Lipschitz mapping. For $\|x\| \ge 2$ this is clear, since $T(x,a) = x$. The restriction of T to the set $\{(x,a) : \|x\| \le 2\}$ is the composition of a Lipschitz mapping

$$(x,a) \mapsto (x, d_a(\|x\|), d_{a^2}(\|x\|), \dots, d_{a^n}(\|x\|)),$$

with

$$(x,t_1,\dots,t_n) \in \{(x,t_1,\dots,t_n) : \|x\| \le 2, 0 \le t_n \le \dots \le t_1 \le 1\} \mapsto t_n^2 U^{[n+1]}_{2\pi/t_1,\dots,2\pi/t_n} x;$$

the latter is 1-Lipschitz in x, and each entry of the matrix

$$t_n^2 U^{[n+1]}_{2\pi/t_1,\dots,2\pi/t_n}$$

is a combination of $\sin\frac{2\pi}{t_i}$ and $\cos\frac{2\pi}{t_i}$, multiplied by t_n^2; as $t_n^2 \le t_i^2$, such an expression has bounded partial derivatives in t_i.

Let us begin the proof of the co-uniformity of T with the following Lemma.

Lemma 1. *For $0<\rho<1$ there exists a constant c_ρ depending only on ρ and n, such that*

$$T(B_\rho(x),a)\supset B_{c_\rho}(T(x,a)),\ \textit{if either } a^n>\rho \textit{ or } \|x\|>1+\rho.$$

Proof. Note that for each nonzero a the inverse of the mapping

$$f_a(x)=T(x,a)\colon \mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$$

can be obtained as

$$f_a^{-1}(y)=\tfrac{p_a(\|y\|)}{\|y\|}\big(U^{[n+1]}_{2\pi/d_a(p_a(\|y\|)),\dots,2\pi/d_{a^n}(p_a(\|y\|))}\big)^{-1}y,$$

where $p_a(t)$ is the inverse of $q_a(t)=td^2_{a^n}(t)$ (the above holds also for $a=0$ as long as $\|x\|>1$). For $t\in(0,1)\cup(1,2)\cup(2,\infty)$, the derivative of $q_a(t)$ is bounded below by $d^2_{a^n}(t)$, i.e. is not less than $a^{2n}\wedge 1$; moreover, $d^2_{a^n}(t)$ is bounded below by ρ^2, when $t>1+\rho$. Thus, if either $a^n\ge\rho>0$ or $\|x\|\ge 1+\rho$, the derivative $p'_a(\|y\|)$ is not greater than $\frac{1}{\rho^2}$ for $y=f_a(x)$. Let us compute the ith partial derivative of f_a^{-1} at $y=f_a(x)$; note that $p_a(\|y\|)=\|x\|$:

$$\begin{aligned}(2)\quad \tfrac{\partial f_a^{-1}(y)}{\partial y_i}&=\tfrac{p'_a(\|y\|)y_i}{\|y\|^2}U(p_a(\|y\|))y-\tfrac{p_a(\|y\|)y_i}{\|y\|^3}U(p_a(\|y\|))y\\&\quad+\tfrac{p_a(\|y\|)}{\|y\|}U(p_a(\|y\|))e_i+\tfrac{p_a(\|y\|)}{\|y\|}p'_a(\|y\|)\tfrac{y_i}{\|y\|}\cdot U'(p_a(\|y\|))y,\end{aligned}$$

where $U(t)$ stands for $(U^{[n+1]}_{2\pi/d_a(t),\dots,2\pi/d_{a^n}(t)})^{-1}$. The norm of the first summand is less than or equal to $\frac{1}{\rho^2}$, the norm of the second is less than or equal to $\frac{p_a(\|y\|)}{\|y\|}=\frac{1}{d^2_{a^n}(\|x\|)}\le\frac{1}{\rho^2}$, the norm of the third is less than or equal to $\frac{p_a(\|y\|)}{\|y\|}\le\frac{1}{\rho^2}$. If $t=p_a(\|y\|)\ge 2$ then $U'(t)=0$, therefore the norm of the fourth summand is less than or equal to $\frac{2}{\|y\|}\frac{1}{\rho^2}\|U'(t)\|\|y\|$. It remains to estimate the norm of the matrix $\|U'(t)\|$. The matrix $(U^{[n+1]}_{\alpha_1,\dots,\alpha_n})^{-1}$ is the product of 2^n-1 rotations in 2-dimensional planes by $\pm\alpha_i$; the derivative of such a rotation with respect to α_j is either zero (if $i\ne j$) or an orthogonal matrix, so $\|\frac{\partial}{\partial\alpha_j}(U^{[n+1]}_{\alpha_1,\dots,\alpha_n})^{-1}\|\le 2^n-1$. Therefore

$$\begin{aligned}\|U'(t)\|\le(2^n-1)\sum_{j=1}^{n}|(\tfrac{2\pi}{d_{a^j}(t)})'|\le 2\pi(2^n-1)\sum_{j=1}^{n}\tfrac{d'_{a^j}(t)}{d^2_{a^j}(t)}\\ \le\tfrac{2\pi(2^n-1)n}{d^2_{a^n}(t)}\le\tfrac{C}{\rho^2},\end{aligned}$$

as $d_{a^n}(t)\le d_{a^j}(t)$ and $d'_{a^j}(t)\le 1$. Thus, the last summand in the right-hand side of (2), as well as the whole gradient of f_a^{-1} at the point $f_a(x)$, has norm not greater than $\frac{c}{\rho^4}$ for some c depending on n.

We have proved an intermediate result: if either $a^n > \rho$ or the norm $\|p_a(y)\| > 1+\rho$, then $\|\nabla f_a^{-1}(y)\| \le c\rho^{-4}$ for some constant $c \ge 1$ depending only on n.

Now in the case $a^n \ge \rho$ the norm of the gradient of $f_a^{-1}(y)$ is bounded by the same constant $c\rho^{-4}$ at all the points y, so the preimage $f_a^{-1}(B_{\rho^5/c}(f_a(x)))$ is contained in $B_\rho(x)$, which is equivalent to $T(B_\rho(x), a) \supset B_{\rho^5/c}(T(x,a))$.

Let us examine the other case: $\|x\| \ge 1+\rho$. Note that

$$q_a(\|x\|) - q_a(1+\tfrac{\rho}{2}) \ge \tfrac{\rho}{2} \min_{\xi \ge 1+\rho/2} q_a'(\xi) \ge (\tfrac{\rho}{2})^3.$$

Therefore for all $z \in B_{\rho^5/16c}(f_a(x))$ we have $\|z\| \ge \|f_a(x)\| - \rho^5/16c \ge \|f_a(x)\| - \rho^3/8 \ge q_a(1+\frac{\rho}{2})$, so the norm of the gradient of f_a^{-1} at z is bounded above by $\frac{16c}{\rho^4}$, as $p_a(\|z\|) \ge 1+\frac{\rho}{2}$. This means that $f_a^{-1}(B_{\rho^5/16c}(f_a(x))) \subset B_\rho(x)$, which is equivalent to $T(B_\rho(x), a) \supset B_{\rho^5/16c}(T(x,a))$. □

Now let us show that T is co-uniform. We may consider only (x,a) in $\mathbb{R}^{n+2}$ with $a \ge 0$ and assume that the radius r lies between 0 and 1.

First case. $r \le 2^{n+9}a$ or $\|x\| > 1 + (\frac{r}{2^{n+9}})^n$. Let $\rho = (\frac{r}{2^{n+9}})^n$ then Lemma 1 implies that

$$TB_r(x,a) \supset T(B_\rho(x), a) \supset B_{c_\rho}(T(x,a)).$$

Second case. $r > 2^{n+9}a$ and $\|x\| \le 1$. Let us show that the set

$$\{T(\tfrac{c}{\gamma^{2n}}y, \gamma) \mid \tfrac{1}{k+1} \le \gamma \le \tfrac{1}{k}, \|y\| = \|x\|, \|y-x\| \le \tfrac{r}{4}\}$$

coincides with the sphere $S_{c\|x\|}$ of radius $c\|x\|$, centered at zero, whenever $k \ge \frac{2^{n+5}}{r}$ is an integer and $\frac{1}{(k+2)^{2n}} \le c \le \frac{1}{(k+1)^{2n}}$.

Take $z \in \mathbb{R}^{n+1}$ of norm $c\|x\|$. Fix $\varphi_1, \dots, \varphi_n \in [0, 2\pi]$ such that $U^{[n+1]}_{\varphi_1,\dots,\varphi_n} x = \frac{z}{c}$. The following lemma will be proved later:

Lemma 2. *For any $\varphi_1, \varphi_2, \dots, \varphi_n \in [0, 2\pi]$ and any positive integer $k \ge 2$ there exists $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ such that*

$$\|U^{[n+1]}_{\varphi_1,\varphi_2,\dots,\varphi_n} x - U^{[n+1]}_{\frac{2\pi}{\gamma},\frac{2\pi}{\gamma^2},\dots,\frac{2\pi}{\gamma^n}} x\| \le \frac{2^{n+1}\pi}{k} \tag{3}$$

for all x: $\|x\| \le 1$.

Now find $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$, such that (3) holds. Then

$$\tfrac{z}{c} \in B_{2^{n+1}\pi/k}(U^{[n+1]}_{\frac{2\pi}{\gamma},\frac{2\pi}{\gamma^2},\dots,\frac{2\pi}{\gamma^n}}(x)) = U^{[n+1]}_{\frac{2\pi}{\gamma},\frac{2\pi}{\gamma^2},\dots,\frac{2\pi}{\gamma^n}} B_{2^{n+1}\pi/k}(x),$$

i.e. $\frac{z}{c} = U^{[n+1]}_{\frac{2\pi}{\gamma},\frac{2\pi}{\gamma^2},\dots,\frac{2\pi}{\gamma^n}}(y)$ for some $y \in B_{2^{n+1}\pi/k}(x) \cap S_{\|x\|}$. This means that $z = T(\frac{c}{\gamma^{2n}}y, \gamma)$, $\|y\| = \|x\|$ and $\|y - x\| \le 2^{n+1}\pi/k \le r\frac{2^{n+1}\pi}{2^{n+5}} \le r/4$, which proves the statement.

We have

$$\|(x,a) - (\tfrac{c}{\gamma^{2n}}y, \gamma)\|^2 \le (\|x-y\| + |1 - \tfrac{c}{\gamma^{2n}}|)^2 + |a-\gamma|^2.$$

Now let k run over all integers greater than $\frac{2^{n+5}}{r}$. For each k, let γ run over $[\frac{1}{k+1}, \frac{1}{k}]$, c run over $[\frac{1}{(k+2)^{2n}}, \frac{1}{(k+1)^{2n}}]$ and y run over the set $\{y \mid \|y\| = \|x\|, \|y - x\| \le \frac{r}{4}\}$. For such γ, c and y we have

$$\begin{aligned} 1 - \tfrac{c}{\gamma^{2n}} &= (1 - \tfrac{\sqrt{c}}{\gamma^n})(1 + \tfrac{\sqrt{c}}{\gamma^n}) \le 2(1 - \tfrac{\sqrt{c}}{\gamma^n}) \\ &\le 2(1 - \tfrac{k^n}{(k+2)^n}) \le 4\tfrac{n(k+2)^{n-1}}{(k+2)^n} \le \tfrac{4n}{k} \le \tfrac{4n}{2^{n+5}}r; \end{aligned}$$

since $0 \le a < \frac{r}{2^{n+5}}$, $0 < \gamma \le \frac{1}{k} \le \frac{r}{2^{n+5}}$ we obtain $|a-\gamma|^2 \le (\frac{r}{2^{n+5}})^2$ and thus

$$(\|x-y\| + |1 - \tfrac{c}{\gamma^{2n}}|)^2 + |a-\gamma|^2 \le (\tfrac{r}{4} + \tfrac{4rn}{2^{n+5}})^2 + (\tfrac{r}{2^{n+5}})^2 < r^2.$$

It means that all the points $(\frac{c}{\gamma^{2n}}y, \gamma)$ as above lie in the ball $B_r(x,a)$. Consequently,

$$TB_r(x,a) \supset \bigcup_{0 \le c \le (\frac{r}{2^{n+6}})^{2n}} S_{c\|x\|} = B_{\|x\|r^{2n}/(2^{n+6})^{2n}}(0), \tag{4}$$

as c runs over $[0, (\frac{r}{2r+2^{n+5}})^{2n}] \supset [0, (\frac{r}{2^{n+6}})^{2n}]$. Note that formula (4) holds for all x, a, r such that $0 \le a < \frac{r}{2^{n+5}}$ and $\|x\| \le 1$.

Since

$$\|T(x,a)\| = a^{2n}\|x\| \le (\tfrac{r}{2^{n+9}})^{2n}\|x\| \le \tfrac{r^{2n}\|x\|}{4(2^{n+6})^{2n}},$$

we conclude that

$$TB_r(x,a) \supset B_{\|x\|r^{2n}/(2^{n+9})^{2n}}(T(x,a)).$$

Now if $\|x\| \ge r/2$ then

$$TB_r(x,a) \supset B_{r/2 \cdot r^{2n}/(2^{n+9})^{2n}}(T(x,a)) = B_{\frac{r^{2n+1}}{2(2^{n+9})^{2n}}}(T(x,a)),$$

while if $\|x\| < r/2$ then, putting $y = rx/(2\|x\|)$,

$$\begin{aligned} TB_r(x,a) &\supset TB_{r/2}(rx/(2\|x\|), a) = TB_{r/2}(y,a) \\ &\supset B_{\|y\|(r/2)^{2n}/(2^{n+6})^{2n}}(0) = B_{\frac{r^{2n+1}}{2^{2n+1}(2^{n+6})^{2n}}}(0) \supset B_{\frac{r^{2n+1}}{2^{2n+2}(2^{n+6})^{2n}}}(z) \end{aligned}$$

for all $\|z\| \le \frac{r^{2n+1}}{2^{2n+2}(2^{n+6})^{2n}}$. Here formula (4) is valid for the triple $y, a, r/2$, since the conditions $0 \le a < \frac{r/2}{2^{n+5}}$ and $\|y\| \le 1$ hold. But

$$\|T(x,a)\| = a^{2n}\|x\| \le (\tfrac{r}{2^{n+9}})^{2n} \cdot r/2 \le \tfrac{r^{2n+1}}{2^{2n+2}(2^{n+6})^{2n}}$$

so $TB_r(x,a) \supset B_{\frac{r^{2n+1}}{2^{2n+2}(2^{n+6})^{2n}}}(T(x,a))$.

THIRD CASE. $r > 2^{n+9}a$ and $1 < \|x\| \le 1 + (\frac{r}{2^{n+9}})^n$. By (4)

$$TB_r(x,a) \supset TB_{r(1-\frac{1}{(2^{n+9})^n})}(\tfrac{x}{\|x\|}, a) \supset B_{(r(1-\frac{1}{(2^{n+9})^n}))^{2n}/(2^{n+6})^{2n}}(0).$$

Now formula (4) is valid since $a < r/2^{n+9} < r(1 - \frac{1}{(2^{n+9})^n})/2^{n+5}$. Since

$$d_{a^n}(\|x\|) \le a^n + \|x\| - 1 \le a^n + (\tfrac{r}{2^{n+9}})^n \le 2 \cdot (\tfrac{r}{2^{n+9}})^n,$$

we obtain

$$\|T(x,a)\| \le (2 \cdot (\tfrac{r}{2^{n+9}})^n)^2\|x\| \le 4\tfrac{r^{2n}}{(2^{n+9})^{2n}}(1 + (\tfrac{r}{2^{n+9}})^n) < \tfrac{1}{2}(r(1 - \tfrac{1}{(2^{n+9})^n})/2^{n+6})^{2n}.$$

Therefore

$$TB_r(x,a) \supset B_{\frac{1}{2}r^{2n}((1-\frac{1}{(2^{n+9})^n})/2^{n+6})^{2n}}(T(x,a)). \qquad \square$$

Remark. One can see that the order of the co-uniformity module $\omega(r)$ at zero varies for different cases: in the first case $\omega(r) \sim r^{5n}$, in the second $\omega(r) \sim r^{2n+1}$ and in the third it is of order r^{2n}.

Proof of lemma 2. Note that the matrix $\frac{\partial}{\partial\varphi_j}U^{[n+1]}_{\varphi_1,\varphi_2,\dots,\varphi_n}$ has operator norm not greater than 2^{j-1}, because it is a sum of 2^{j-1} matrices of norm 1. Therefore

$$\|U^{[n+1]}_{\varphi_1,\varphi_2,\dots,\varphi_n} - U^{[n+1]}_{\widetilde{\varphi}_1,\widetilde{\varphi}_2,\dots,\widetilde{\varphi}_n}\| \le \sum_{j=1}^{n} 2^{j-1}[(\varphi_j - \widetilde{\varphi}_j) \bmod 2\pi].$$

Hence if $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ satisfies (5) below, then for all x such that $\|x\| \le 1$

$$\|U^{[n+1]}_{\frac{2\pi}{\gamma},\frac{2\pi}{\gamma^2},\dots,\frac{2\pi}{\gamma^n}}x - U^{[n+1]}_{\varphi_1,\varphi_2,\dots,\varphi_n}x\| \le \sum_{j=1}^{n-1} 2^{j-1}\tfrac{4\pi}{k} \le \tfrac{2^{n+1}\pi}{k}. \qquad \square$$

Lemma 3. *For any $\varphi_1, \varphi_2, \dots, \varphi_n \in [0, 2\pi]$ and any positive integer $k \ge 2$ there exists $\gamma \in [\frac{1}{k+1}, \frac{1}{k}]$ such that*

$$\text{(5)} \quad \varphi_j - \tfrac{2\pi}{\gamma^j} \bmod 2\pi \le \tfrac{4\pi}{k} \ \textit{for all } j = 1, \dots, n-1 \quad \textit{and } \varphi_n - \tfrac{2\pi}{\gamma^n} \bmod 2\pi = 0.$$

Proof. Let $N(j) = (k+1)^j - k^j - 1$. We define the sequence $\{a_m^{[n]}\}_{m=0}^{N(n)}$ by

$$a_m^{[n]} = 2\pi(k^n + m) + \varphi_n.$$

Now for each $j = n, \ldots, 2$ having constructed the sequence $\{a_m^{[j]}\}_{m=0}^{N(j)}$ such that

$$a_m^{[j]} \in [\ 2\pi(k^j + m), 2\pi(k^j + m + 1)\ ] \text{ and } a_m^{[j]} - \varphi_j \bmod 2\pi \leq \tfrac{4\pi}{k}$$

we construct $\{a_m^{[j-1]}\}_{m=0}^{N(j-1)}$ as follows. Note first that the derivative of the function $q_j(t) = 2\pi(\frac{t}{2\pi})^{\frac{j-1}{j}}$ is less than $\frac{1}{k}$ for $t \in [\ 2\pi k^j, 2\pi(k+1)^j\ ]$. This implies that

$$q_j(a_0^{[j]}) - q_j(2\pi k^j) \leq (a_0^{[j]} - 2\pi k^j)\tfrac{1}{k} \leq \tfrac{2\pi}{k}$$

and, for $0 \leq m \leq N(j) - 1$,

$$q_j(a_{m+1}^{[j]}) - q_j(a_m^{[j]}) \leq (a_{m+1}^{[j]} - a_m^{[j]})\tfrac{1}{k} \leq \tfrac{4\pi}{k}.$$

Also

$$q_j(2\pi(k+1)^j) - q_j(a_{N(j)}^{[j]}) \leq (2\pi(k+1)^j - a_{N(j)}^{[j]})\tfrac{1}{k} \leq \tfrac{2\pi}{k}.$$

It follows that we can choose $\{a_m^{[j-1]}\}_{m=0}^{N(j-1)}$ among $\{q_j(a_m^{[j]})\}_{m=0}^{N(j)}$ so that

$$a_m^{[j-1]} \in [\ 2\pi(k^{j-1} + m), 2\pi(k^{j-1} + m + 1)\ ]$$
$$\text{and } a_m^{[j-1]} - \varphi_{j-1} \bmod 2\pi \leq \tfrac{4\pi}{k}.$$

Consider $\{a_m^{[j]}\}_{m=0}^{N(j)}$ for $j = 1$ — this is one point. Let us define $\gamma = \frac{2\pi}{a_0^{[1]}}$. Then $\frac{2\pi}{\gamma^j}$ belongs to $\{a_m^{[j]}\}_{m=0}^{N(j)}$ for each j, so (5) holds. □

References

[BJLPS] S. Bates, W. B. Johnson, J. Lindenstrauss, D. Preiss, and G. Schechtman, *Affine approximation of Lipschitz functions and nonlinear quotients,* **Geom. Funct. Anal. 9** (1999), 1092–1127.

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis, Volume 1,* Colloquium Publications AMS, vol. 48 (1999).

[G] E. Gorelik, *The uniform nonequivalence of L_p and ℓ_p,* **Israel J. Math. 87** (1994), 1-8.

[JLS] W. B. Johnson, J. Lindenstrauss, and G. Schechtman, *Banach spaces determined by their uniform structures,* **Geom. Funct. Anal. 6** (1996), no. 3, 430—470.

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel, e-mail: maleva@wisdom.weizmann.ac.il