X

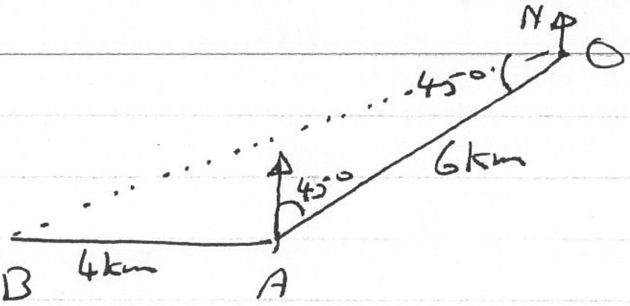
1. Vectors

Quantities which have only magnitude (or size) are Scalar.

Quantities which have magnitude and direction are Vectors.

Example 1.1

Alice walks 6km South-West then 4km due West. How far and in what direction is Alice?



(1) Draw a diagram. - Triangle.
Two stages OA
AB

Goal, (i) find length of OB
(ii) find angle AOB.

[Cosine rule] $OB^2 = OA^2 + AB^2 - 2 \times OA \times AB \cdot \cos \cancel{135°}\ B\hat{A}O$
$= 6^2 + 4^2 - 2 \times 6 \times 4 \cos(135°)$
≈ 9.27 km

[Sine rule] $\dfrac{OB}{\sin O\hat{A}B} = \dfrac{AB}{\sin B\hat{O}A}$

$$\frac{9.27}{\sin 135} = \frac{4}{\sin B\hat{O}A}$$

$B\hat{O}A = 17.77°$, hence $B\hat{O}S = 45° + 17.17° = 62.77°$.
$+180°$ $= 242.77°$ N

X

1.1 Vector representations

In example 1.1 the answer is a line segment of a given <u>length</u> in a particular <u>direction</u>.

To distinguish the <u>length</u> OA from <u>vector</u> OA

we write length OA, or $|OA|$ for emphasis

Vector $\overrightarrow{OA}$.

So, for Alice $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

The result of adding two vectors is the <u>resultant</u>.

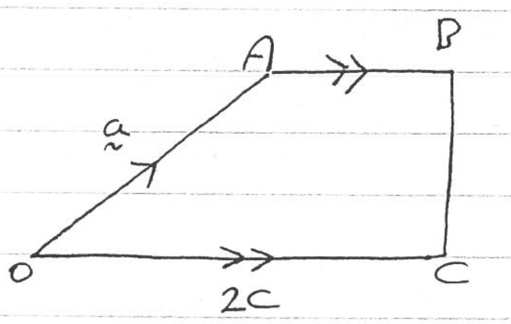
<u>Note</u> (1) $\boxed{\text{Direction matters!}}$

$$\overrightarrow{OA} \neq \overrightarrow{AO}. \qquad (\text{but } |OA| = |AO|).$$

(2) We might use a single letter, with a squiggle, to represent a vector, e.g. $\underset{\sim}{x}$.

$$\text{If } \overrightarrow{OA} = \underset{\sim}{x} \text{ then } \overrightarrow{AO} = -\underset{\sim}{x}.$$

<u>Example 1.2</u>



$\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OC} = \cancel{2a}\ 2\underset{\sim}{c}$

~~OC // A~~

$\overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$ and half as long.

So $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}2\underset{\sim}{c} = \underset{\sim}{c}$.

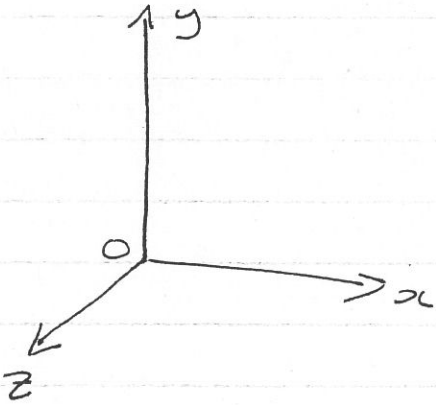
$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \underset{\sim}{a} + \underset{\sim}{c}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{OC} \\ &= -(\overrightarrow{AB}) - \overrightarrow{OA} + \overrightarrow{OC} \\ &= -\underset{\sim}{c} - \underset{\sim}{a} + 2\underset{\sim}{c} \quad = \underset{\sim}{c} - \underset{\sim}{a}.\end{aligned}$$

1.2 Unit Vectors

If $AB = 1$, ie magnitude is one, then AB is a <u>Unit vector</u>.

Unit vectors may be in any direction.



Given a rectangular $x, y, (z)$ coordinate system in 2 (or 3) dimensions.

$\underset{\sim}{i}$ is a unit vector parallel to the x-axis in the positive direction

$\underset{\sim}{j}$ is a Unit Vector parallel to the y-axis in the positive direction

When in 3 dimensions,

$\underset{\sim}{k}$ is a unit vector parallel to the z-axis in a positive direction

Adding vectors written in terms of unit vectors is straightforward.

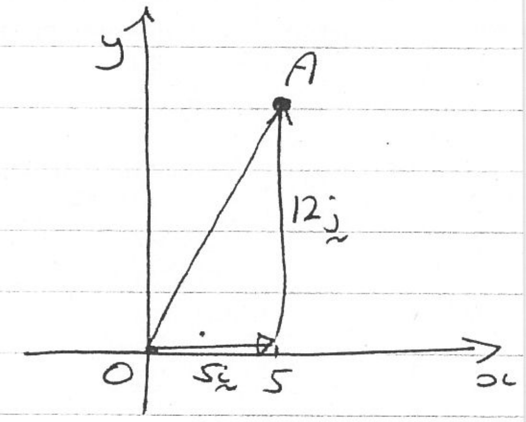
Example 1.3

$\underset{\sim}{a} = (3\underset{\sim}{i} + 2\underset{\sim}{j})$, $\underset{\sim}{b} = (5i - 6j)$

the resultant of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is

$$\underset{\sim}{a} + \underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{i} - 6\underset{\sim}{j}$$
$$= 8\underset{\sim}{i} - 4\underset{\sim}{j}.$$

Example 1.4



Find the magnitude of $5\underset{\sim}{i} + 12\underset{\sim}{j}$.

$OA^2 = 5^2 + 12^2$

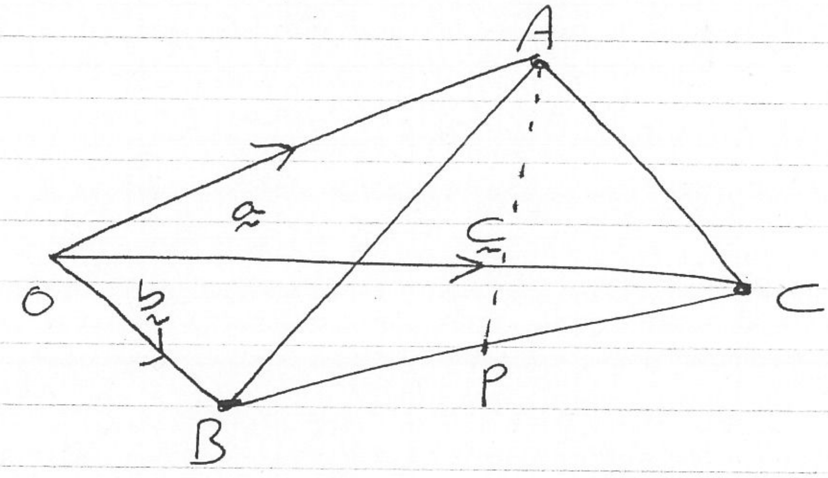
$OA = 13.$

Example 1.5. Centroid of a triangle

Show that the medians of any triangle meet in a point (called the centroid) which divides each of them in a ratio 2:1.

Let triangle ABC, where $\vec{OA} = a$
$\vec{OB} = b$
$\vec{OC} = c.$

The median is the segment joining a vertex to the midpoint of the opposite side. X



P is the midpoint of BC. $\vec{OP} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{p}$

The point X, which divides AP in ratio 2:1 has position vector

$$\vec{OX} = \underset{\sim}{x} = \frac{\underset{\sim}{a} + 2\underset{\sim}{p}}{2+1} = \frac{\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}}{3}.$$

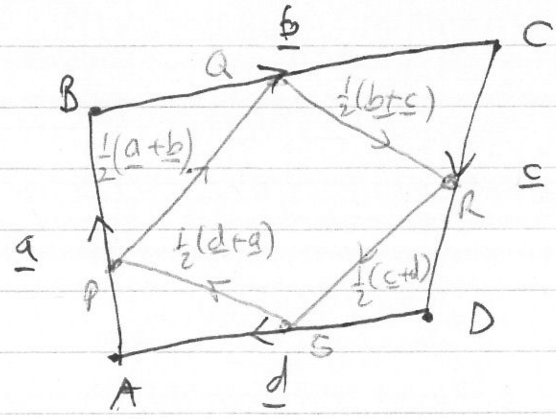
Repeat this to find the point Y & Z, which divide B-(AC) and C(BA) in ratio 2:1 respectively. [Ex].

Since X = Y = Z, these lines meet at a point.

Example 1.6

Suppose that the midpoint of consecutive sides of a quadrilateral are connected by straight lines. Prove that the resulting quadrilateral is a parallelogram.

SOLUTION:



Let $ABCD$ be the given quadrilateral and P, Q, R & S the midpoints of its sides as shown here.

Then
$$\vec{PQ} = \tfrac{1}{2}\underline{a} + \tfrac{1}{2}\underline{b}$$
$$\vec{QR} = \tfrac{1}{2}\underline{b} + \tfrac{1}{2}\underline{c}$$
$$\vec{RS} = \tfrac{1}{2}\underline{c} + \tfrac{1}{2}\underline{d}$$
$$\vec{SP} = \tfrac{1}{2}\underline{d} + \tfrac{1}{2}\underline{a}$$

Also we have $\underline{a}+\underline{b}+\underline{c}+\underline{d} = 0$ so

$$\vec{PQ} = \tfrac{1}{2}(\underline{a}+\underline{b}) = -\tfrac{1}{2}(\underline{c}+\underline{d}) = \vec{SR}$$ and

$$\vec{QR} = \tfrac{1}{2}(\underline{b}+\underline{c}) = -\tfrac{1}{2}(\underline{d}+\underline{a}) = \vec{PS}$$

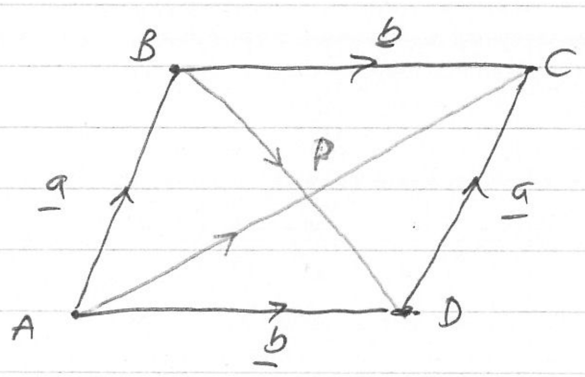
Thus opposite sides are of equal magnitude and are parallel so this $PQRS$ is a parallelogram.

Example 1.7

Prove that the diagonals of a parallelogram bisect each other.

SOLUTION:

Let ABCD be the given parallelogram below



where P is location where the diagonals intersect.

We have $\overrightarrow{BD} = \underline{b} - \underline{a}$ so $\overrightarrow{BP} = x(\underline{b} - \underline{a})$ for some $x \in (0,1)$ and

$\overrightarrow{AC} = \underline{a} + \underline{b}$ with $\overrightarrow{AP} = y(\underline{a} + \underline{b})$ for some $y \in (0,1)$

Now $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{AP} - \overrightarrow{BP}$

$\Rightarrow$

$$\begin{aligned} \underline{a} &= y(\underline{a} + \underline{b}) - x(\underline{b} - \underline{a}) \\ &= (x+y)\,\underline{a} + (y-x)\,\underline{b} \end{aligned}$$

Now $\underline{a}$ and $\underline{b}$ are non-colinear (i.e. not parallel) so

$y - x = 0$ or $x = y$ and $x + y = 1$

Example 1.8

Prove that two vectors $\underline{a}$ and $\underline{b}$ must have equal magnitudes if their sum is perpendicular to their difference

SOLUTION:

We have that $\underline{a}+\underline{b}$ is perpendicular to $\underline{a}-\underline{b}$

so

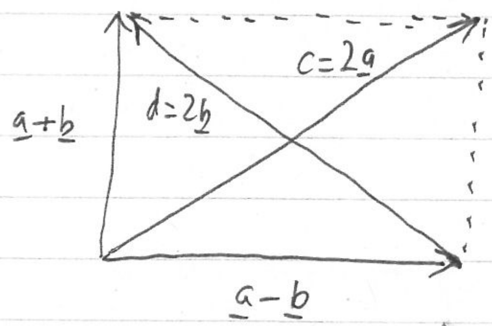
$$\underline{a}+\underline{b} \perp \underline{a}-\underline{b}$$

their sum is

We have that, $(\underline{a}+\underline{b}) + (\underline{a}-\underline{b}) = 2\underline{a}$

and $(\underline{a}+\underline{b}) - (\underline{a}-\underline{b}) = 2\underline{b}$

We can illustrate this as



$\underline{c}$ and $\underline{d}$ are two diagonal lines of a rectangle so

$|\underline{c}| = |\underline{d}|$ or

$2|\underline{a}| = 2|\underline{b}|$ or

$|\underline{a}| = |\underline{b}|$

as required.

Example 1.9

In a methane molecule, CH_4, each hydrogen atom is at the corner of a tetrahedron with the Carbon atom at the center. In a coordinate system centred on the carbon atom, if the direction of one of the C–H bonds is described by the vector

$\underline{A} = \underline{i} + \underline{j} + \underline{k}$ and the direction of the adjacent C–H bond by the vector

$\underline{B} = \underline{i} - \underline{j} - \underline{k}$. Calculate the angle between these two bonds.

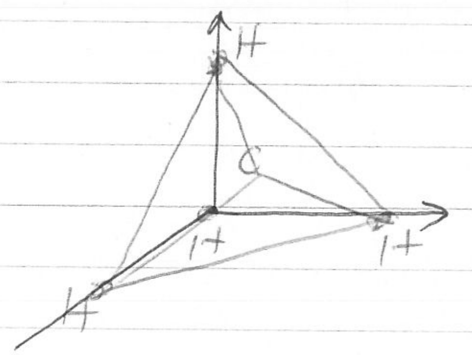
SOLUTION:

The angle between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\underline{a}\cdot\underline{b}}{|a|\,|b|}\right)$$ so in this case we have

$$\theta = \cos^{-1}\left(\frac{(1)(1) + (1)(-1) + 1(-1)}{2\sqrt{3}}\right)$$

$$= 106.7^\circ$$



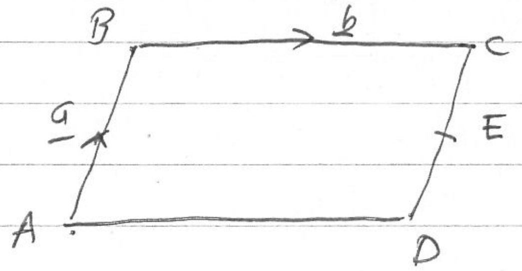
Problems in Mechanics. Week 1

Q1 : The diagram below shows a parallellogram with $\vec{AB} = \underline{a}$ and $\vec{BC} = \underline{b}$. E is the mid point of CD. Express the following vectors in terms of $\underline{a}$ and $\underline{b}$

(a) $\vec{AD}$ (b) $\vec{DC}$ (c) $\vec{CD}$ (d) $\vec{DE}$ (e) $\vec{AE}$

SOLUTION :

Let ABCD be given below



(a) $\vec{AD}$ is the same length as $\vec{BC}$ and in the same direction so $\vec{AD} = \vec{BC} = \underline{b}$

(b) In a similar way

$$\vec{DC} = \underline{a}$$

(c) $\vec{CD}$ is the same as $\vec{AB}$ but in opposite direction so

$$\vec{CD} = -\underline{a}$$

(d) $\vec{DE} = \frac{1}{2}\vec{DC} = \frac{1}{2}\underline{a}$

(e) $\vec{AE} = \vec{AD} + \vec{DE}$
$= \underline{b} + \frac{1}{2}\underline{a}$

Q2 : This question is about determining the angle between each CH bond in a CH_4 (methane) molecule. Assume we have a coordinate system with the origin located at the Carbon (C) ~~molecule~~ atom. Now let $\underline{r}_i$ denote the location of Hydrogen (H) atoms for $i = 1, 2, 3, 4$ as there are 4 Hydrogen atoms in methane. If the angle between each C-H bond is the same and they are the same distance apart and using the symmetry relation

$$\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4 = 0$$ show that the

angle between them (i.e the C-H bonds) is

$$\theta = 109.47^\circ \quad \text{or} \quad \cos^{-1}(-1/3)$$

SOLUTION:

We know that

$$\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4 = 0$$

Since each of these vectors has the same magnitude, say, α we can take the scalar (or dot) product with any $\underline{r}_i$ (say $\underline{r}_2$) to get

$$\underline{r}_2 \cdot (\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4) = 0$$

$\Rightarrow$

$$r^2 + 3r^2 \cos\theta = 0$$

$$r^2(1 + 3\cos\theta) = 0 \quad \text{or} \quad \theta = \cos^{-1}\left(\tfrac{1}{3}\right).$$

2. Distance, velocity and acceleration.

A particle, P, has position vector $\underset{\sim}{x}$ at time t, relative to some origin O.

$$\text{mean velocity} = \frac{\text{displacement}}{\text{time}}.$$

$$\boxed{\underset{\sim}{v} = \frac{\underset{\sim}{x}(t_2) - \underset{\sim}{x}(t_1)}{t_2 - t_1}}$$

– a vector

The instantaneous velocity is the limit as $t_2 \to t_1$.

$$\underset{\sim}{v}(t_1) = \lim_{t_2 \to t_1} \left(\frac{\underset{\sim}{x}(t_1) - \underset{\sim}{x}(t_1)}{t_2 - t_1} \right)$$

So $$\underset{\sim}{v}(t) = \frac{d\,\underset{\sim}{x}(t)}{dt}.$$

$\boxed{\text{The speed is } |\underset{\sim}{v}|.}$

Velocity is rate of change of position.

Accelleration is rate of change of velocity

$$\boxed{\underset{\sim}{a} = \frac{d\,\underset{\sim}{v}(t)}{dt}}$$

2.1 Uniform acceleration in one dimension

Assume a particle P moves in one dimension under constant acceleration a.

Conventions

u = initial velocity $\quad v$ = final velocity

a = acceleration (constant)

t = time

s = displacement

By definition $\quad a = \dot{v}(t)$

So

$$\int_0^t a\,dt = \int_0^t \dot{v}(t)\,dt$$

$$\therefore \quad a(t-0) = \left[v(t)\right]_{t=0}^{t=t} = v(t) - v(0)$$

So $\quad at = v - u$, or $\boxed{a = \frac{v-u}{t}}$ or $\boxed{v = u + at}$ (2.1)

We assumed acceleration is constant.

$$\text{average velocity} = \frac{u+v}{2}$$

$$= \frac{\text{displacement}}{\text{time}} = \frac{s}{t}$$

So $\quad \frac{s}{t} = \frac{u+v}{2}$, $\quad \boxed{s = \frac{u+v}{2}t}$ (2.2)

Substitute v from (2.1) into (2.2)

$$\boxed{s = ut + \frac{1}{2}at^2} \qquad (2.3)$$

Substitute t from (2.1) into (2.2).

$$t = \frac{v-u}{a} \quad \text{gives} \quad s = \frac{u+v}{2} \cdot \frac{v-u}{a}$$

So $\quad 2as = v^2 - u^2$

or $\quad \boxed{v^2 = u^2 + 2as} \qquad (2.4)$

(2.1) – (2.4) apply for constant acceleration and are worth remembering.

Example 2.1   Use example enviroment?

A car accelerates from rest to 100 km/h in 12 seconds. Find (i) acceleration (assumed constant)
(ii) distance travelled.

(i) $\quad 100\,\text{km/h} = \frac{100 \times 1000}{60^2}\,\text{m/s} = \frac{1000}{36}\,\text{m/s}.$

So $u = 0$, $v = \frac{1000}{36}$, $t = 12$.

$v = u + at$, $\quad$ so $\quad a = \frac{1000}{36 \times 12} \approx 2.31\,\text{m s}^{-2}.$

(ii) $\quad s = \frac{u+v}{2} \cdot t = \frac{1}{2} \cdot \frac{1000}{36} \cdot 12 \approx 166\text{m}$

$$s = ut + \frac{1}{2}at^2 = \frac{1}{2} \cdot 2.31 \times 12^2 \approx 166\text{m}.$$

2.2 Free fall under gravity

In many situations acceleration due to gravity can be modelled as constant.

Near the earth's surface $g \approx 9.8\,m\,s^{-2}$
$[\approx 10\,m s^{-2}]$.

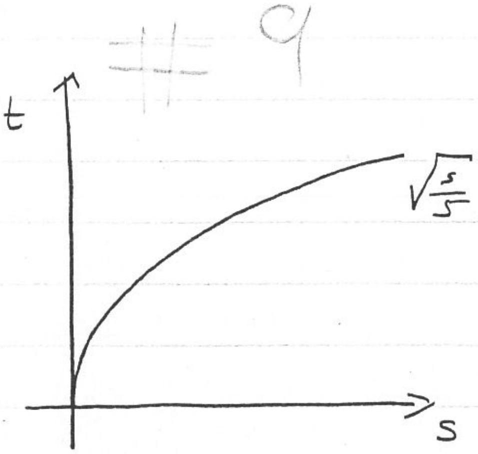
Example 2.2

A body accelerates at $10ms^{-2}$. How long does it take to move 5m?, 10m? 15m? 20m.

$$S = ut + \tfrac{1}{2}at^2 \qquad u = 0, \quad a = 10$$

So $t = \sqrt{\frac{2S}{10}} = \sqrt{\frac{S}{5}}$.

S	$\sqrt{S/5} = t$
5m	1s.
10	1.4s
15	1.7s
20	2s.



Example 2.3

A ball is thrown vertically upwards with a velocity of 6.86 ~~6.72~~ ~~68.6~~ ms from a platform ~~5.76~~ 5.88 m heigh.
(i) When will it hit the ground?

$u =$ ~~6.72~~ ~~68.6ms~~ 6.86 m/s
$a = -9.8ms^{-2}$
$S =$ ~~+57.76m~~ 5.88 m/s

$s = ut + \frac{1}{2} at^2$

$-5.88 = 6.86t - \frac{1}{2} \cdot 9.8 t^2.$

$4.9t^2 - 6.86t - 5.88 = 0$

$t^2 - 1.4t - 1.2 = 0$

$5t^2 - 7t - 6 = 0$

$(t-2)(5t+3) = 0$

$\underline{t = 2}$ or $t = \frac{-3}{5}$.

(ii) What is its maximum height from the floor?

$V^2 = u^2 + 2as$

$0 = 6.86^2 - 2 \times 9.8 \times s$

$s = \frac{6.86^2}{2 \times 9.8} \approx 2.4\text{m}$

Height above the floor is $2.40 + 5.88$

$= 8.28\text{ m}.$

3. Force and Newton's Laws

What causes a body to move/accelerate?

[A profound philosophical question!].

Newton's Laws

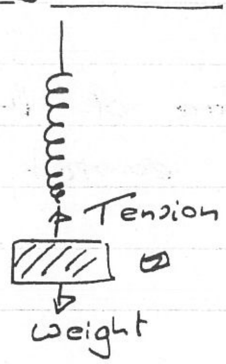
[Remember them!]

> 1. A change in the state of motion is caused by a force.

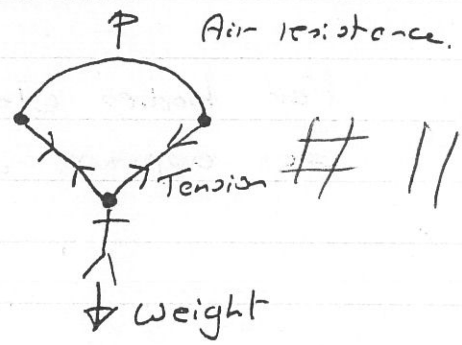
A body at rest or with constant velocity has no overall force.

Spring & mass (rest)

#10



Parachute ~~Glider~~ (constant glide)



#11

> **Newton's first law of motion**
>
> N1. A body will remain at rest, or will continue to move with a constant velocity, unless an external force causes it to do otherwise

> **Newton's 2nd law of motion**
>
> N2. Acceleration is caused by resultant force, and is proportional to that force
>
> $$\underset{\sim}{F} = m\underset{\sim}{a}. \qquad (3.1)$$

Notice (i) that <u>mass</u> is a property of the body.

(ii) this is a <u>Vector</u> equation.

Unhappy w/ this.

(iii) mass, kilograms kg
$\underset{\sim}{a}$ meters per second squared, ms^{-2}
F, <u>Newton</u>.

Gravity and Weight

A falling body undergoes an acceleration of $g = 9.8\,ms^{-2}$.

This must be caused by a <u>force</u>.
This force is called <u>Weight</u>.

A body of mass m, has weight mg.

<u>Earth</u> $g = 9.8\,ms^{-2}$ <u>Moon</u> $g \approx 1.6\,ms^{-2}$

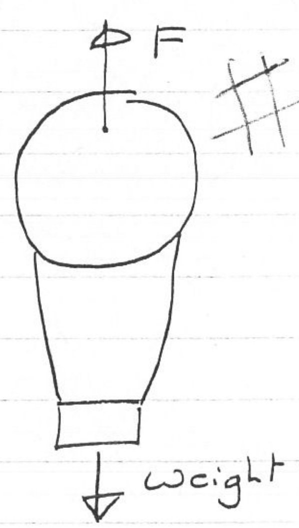
Person, mass 70kg

Weight $= 70 \times 9.8$
$= 686\,N$.

Weight $= 70 \times 1.6$
$= 112\,N$.

Example, Balloon

#13



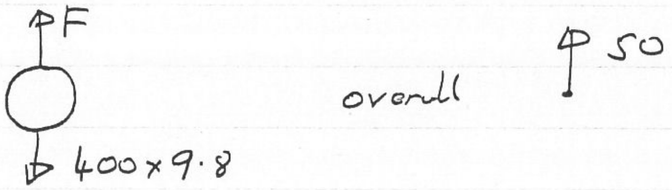
A balloon rises from the ground with uniform accelleration.
After 20 s, the height is 25m.
Total mass is ~~350 kg.~~ 400 kg
Find the lifting force.

Using (2·3) $s = ut + \frac{1}{2}at^2$

$$25\,\text{m} = 0 + \tfrac{1}{2} \cdot a \cdot 20^2$$

$$\therefore \quad a = 0 \cdot 125 \ \text{m s}^{-2}.$$

$F = ma$, so overall force is 50 N.

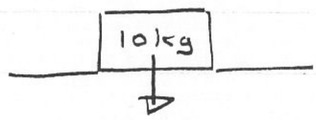


#14

So $F = 50 + 400 \times 9.8 = 3970$ N.

<u>Newton's third law</u> Every action has an equal and opposite reaction.

<u>Example:</u> ~~Balloon~~. Contact



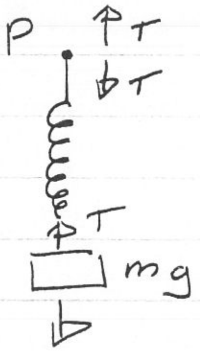
$10 \times 9.8 = 98$ N.

#15

Weight of a box in contact with a table is 98 N.

This is in equilibrium - no acceleration.
So table exerts a force of 98 N on the box.

<u>Example</u> Spring at equilibrium.

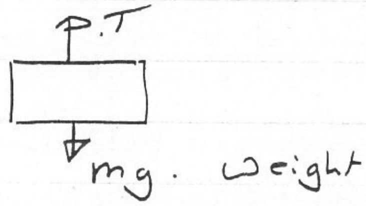


#16

~~Forces acting on the mass:~~
~~Weight = mg~~
~~Tension = T = mg (equilibrium)~~

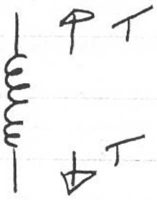
~~Force acting on the support P~~
~~Tension T down.~~

Mass m hangs in equilibrium.

#17 

The spring exerts an upward force T on the mass.

Hence the mass exerts a downward, equal, force T on the spring.

#18 

The spring is supported in equilibrium

So the ~~sp~~ support exerts a force T on the spring upwards.

The spring exerts a force T on the pivot.

$\boxed{\text{Always state what a force acts on.}}$

In this case $T = mg$ - equilibrium.

T is called <u>tension</u>.

<u>Connected particles</u>

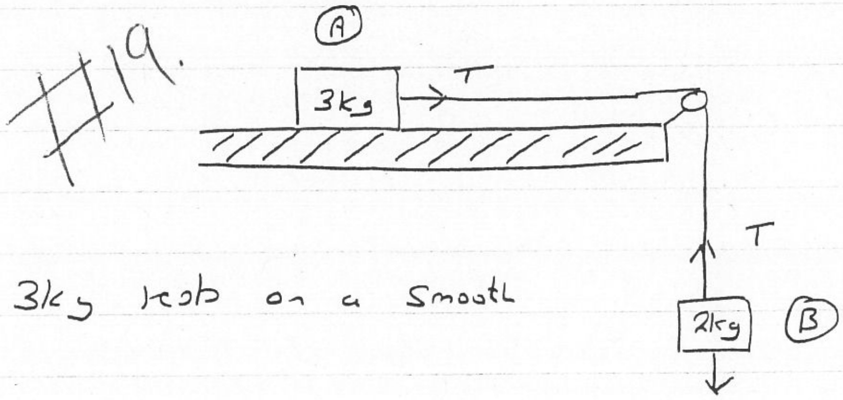
All strings are "light" (have no mass)
"inextensible" (don't stretch)

All pullies and surfaces are "smooth" (no friction).

[Elasticity and friction will be introduced later].

Unless otherwise stated.

Example



Body of mass 3kg rests on a smooth table.

Connected by a light string (no mass) to a mass of 2k hanging freely.

Forces on (A)

Gravity 3g down.
~~Reaction 3g up. (equal and opposite).~~

No vertical acceleration.
So reaction force of table on (A) is 3g.

~~Tension~~ Tension T pulling to the right. F=ma
So

$$T = 3a. \tag{3.2}$$

Forces on (B)

Gravity 2g down.
Tension T up.

$$2g - T = 2a \tag{3.3}$$

Solving (3.2) & (3.3) simultaneously gives, for a,

$$2g - 3a = 2a$$

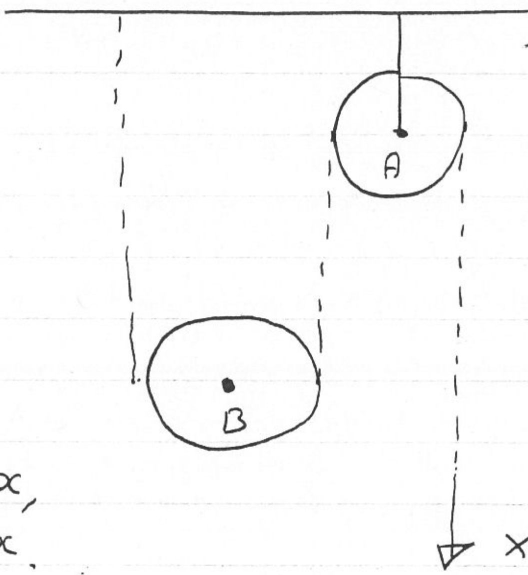
so $a = \frac{2}{5}g$.

Pulley?

Pully Systems

A is fixed.

B may be raised by pulling the string at X.



#20

String vertical.
Pully frictionless
String light
String inextensible.

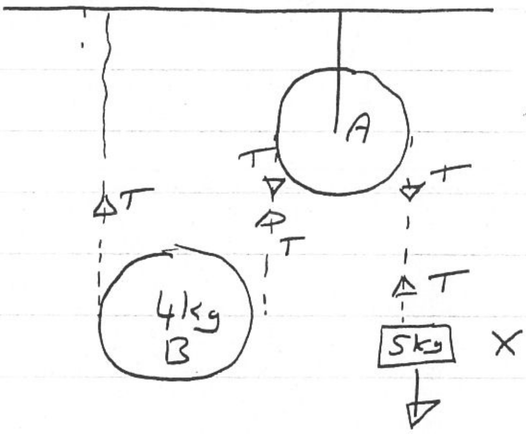
For B to move x, X must move $2x$.

Since $a = \dfrac{d^2x}{dt^2}$, if B accelerates up with acceleration a, then X accelerates down at $2a$.

Example

System released from rest.

a = upward acceleration of B.



#21

Forces on B. $2T - 4g = 4a$.

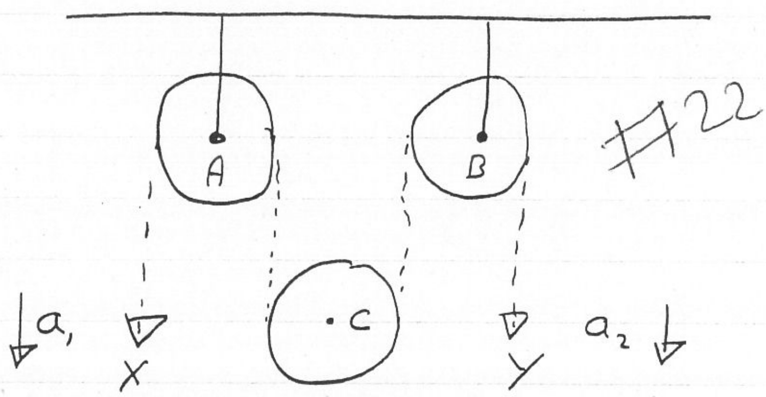
Forces on X $5g - T = 5 \times 2a$

Omitted
Geometric constraint from pully.

If $g = 9.8\ m/s^2$ $a = 2.45$, $T = 24.5$ (Ex).

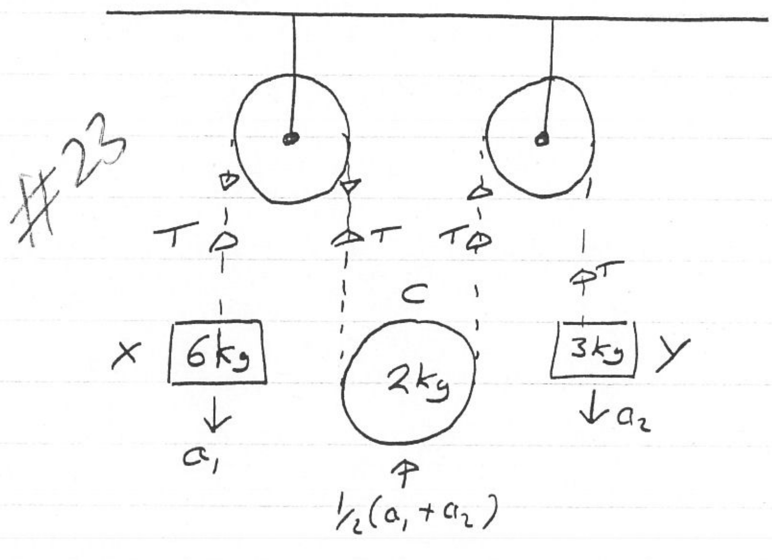
Example

A & B are fixed.
C moveable.



If X moves down by x
Y moves down by y
then C moves up by $\frac{1}{2}(x+y)$
[String betwen A & B is shortened by $\frac{1}{2}(x+y)$].

So If X accelerates down by a_1
Y — " — a_2
C accelerates up by $\frac{1}{2}(a_1+a_2)$.



Forces on X
$$6g - T = 6a_1$$
Forces on Y
$$3g - T = 3a_2$$
Forces on C
$$2T - 2g = 2 \times \tfrac{1}{2}(a_1+a_2)$$

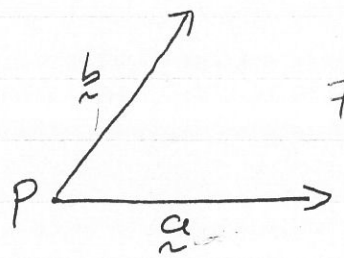
Solving (Ex) $a_1 = \frac{11}{15}g$ $a_2 = \frac{7}{15}g$.

4 Resultants and Components of forces.

So far all forces have been vertical or horizontal.

Forces are vectors. Take two forces $\underset{\sim}{a}$ and $\underset{\sim}{b}$ acting on a point P.

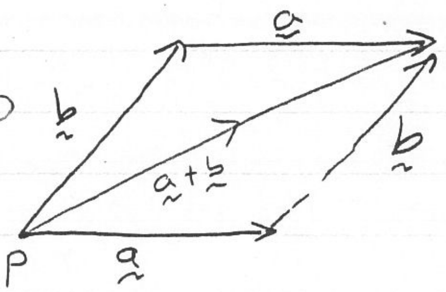
The resultant is $\underset{\sim}{a} + \underset{\sim}{b}$.



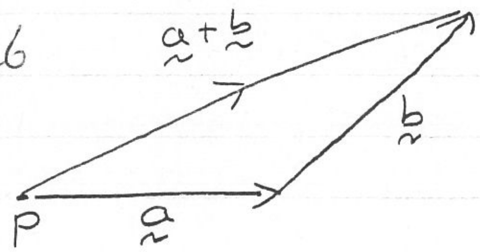
\#24

This can be found with a parallelogram of forces or triangle of forces.

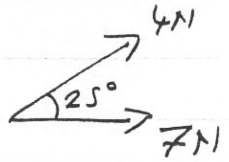
\#25



\#26

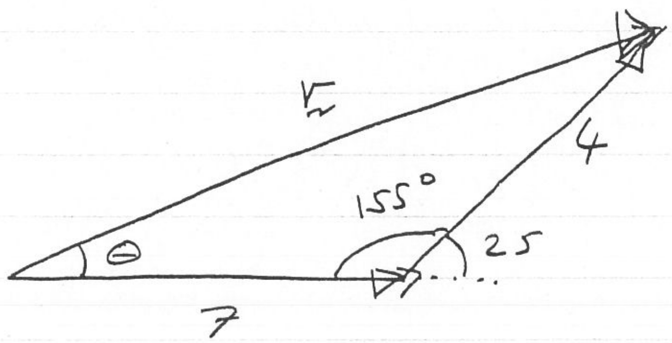


Example Find the resultant of



\#27

Construct a triangle of forces



\#28

By the cosine rule $|\underset{\sim}{r}|^2 = 7^2 + 4^2 - 2 \times 4 \times 7 \times \cos 155°$

so $|\underset{\sim}{r}| = 10.8\,N$.

Direction, θ.

$$\frac{\sin\theta}{4} = \frac{\sin 155°}{10.8}, \quad \text{so } \theta = 9°.$$

Care is needed with angles, directions.

Any number of forces from a single point can be resolved into a single resultant force by vector addition.

Components of a force

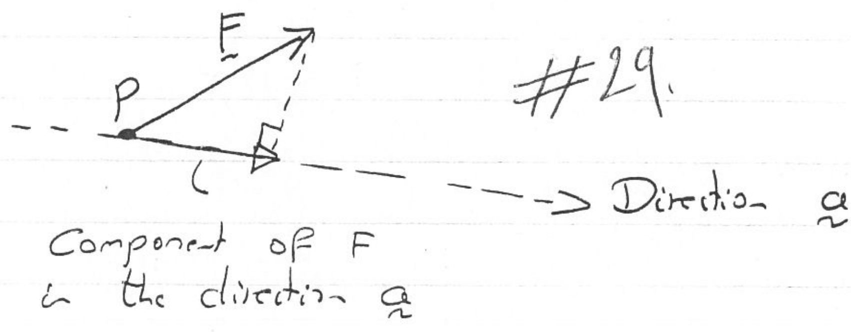
Two forces $\xrightarrow{\text{add}}$ "resultant"

Single force $\xrightarrow{\text{"split"}}$ two (three) components

← Multiple.

Mutually perpendicular directions are very convenient.

Definition. The component of a force $\underset{\sim}{F}$ in the direction $\underset{\sim}{a}$ is the projection of $\underset{\sim}{F}$ onto $\underset{\sim}{a}$.

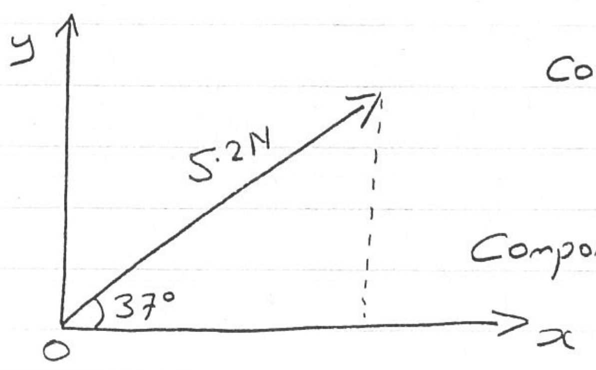


#29.

[Projection: drop a perpendicular from F onto the line in the direction $\underset{\sim}{a}$].

Example

#30



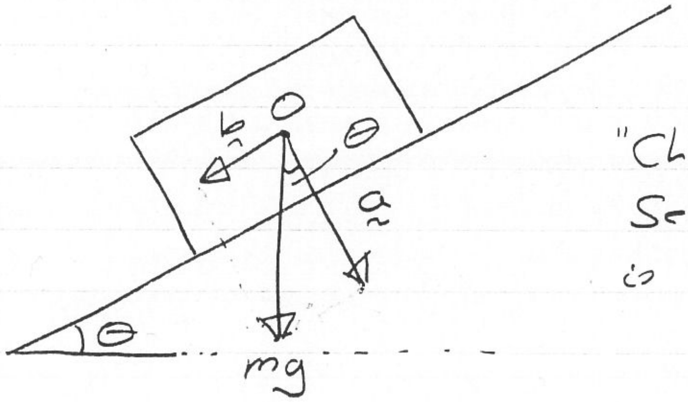
Component in x-direction
$5.2 \times \cos(37°) = 4.15$

Component in y-direction
$5.2 \times \sin(37°) = 3.13N$

Example

A block sits at the top of a smooth (frictionless) slope. Find the force parallel and perpendicular to the slope.

#31



"Chasing angles" we see the marked angle is also θ.

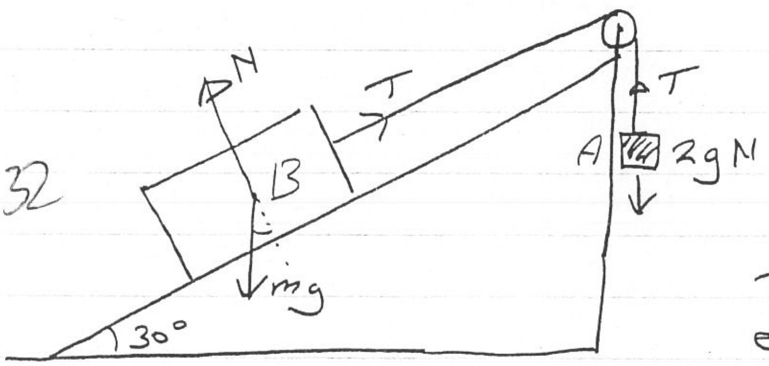
So $\underset{\sim}{a} = mg \cdot \cos\theta$ Perpendicular.
$\underset{\sim}{b} = mg \sin\theta$ Parallel to slope.

[So we could now calculate acceleration along slope].

Example Systems with more than one particle.

A light inextensible string passes over a smooth pulley at the top of a smooth plane, inclined at 30° to the horizontal.

#32



One mass, A, hangs
One mass B, sits on the slope

The system is in equilibrium.

Find T, the tension in the string.
N, the normal (perpendicular) reaction.
m, the mass of B.

The hanging mass is in equilibrium.

Resolving forces on A: $T = 2g$ N.

Resolving forces on B parallel to the slope

$$T = mg.\,\cancel{\cos}\,\sin 30^\circ$$
$$2g = m.g.\,\tfrac{1}{2}$$
$$\text{So} \quad m = 4\,kg.$$

Resolving forces on B perpendicular to the slope.

$$N = m.g.\,\cancel{\sin}\,\cos 30^\circ$$
$$N = 4 \times 9.8 \times \cancel{\sin}\,\cos 30^\circ \approx 33.9\,N.$$

5. Equilibrium & Acceleration under forces

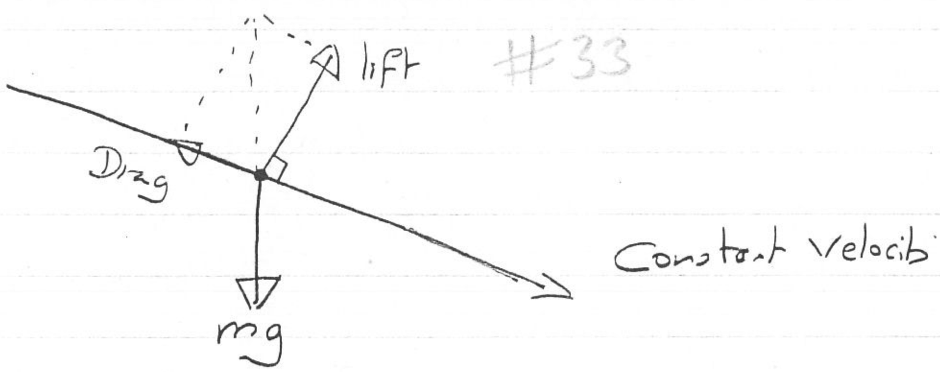
N1: Bodies remain at rest or constant velocity unless a force acts on them.

N2: $F = ma$.

So $F = 0 \not\Rightarrow$ (does not imply) Velocity $= 0$.

Example 5.1

Imagine a paraglider pilot flying in still air. They glide at a constant speed and along a sloped path.



What are the other forces on the pilot?

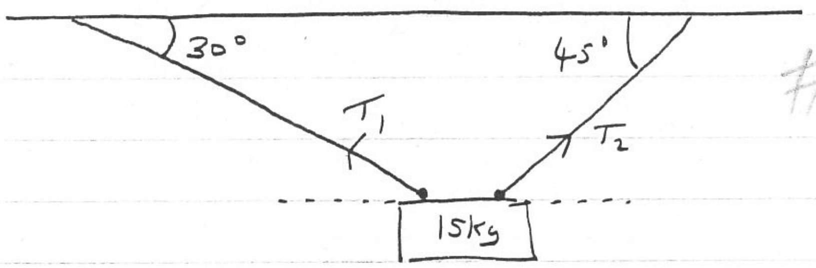
Since there is no acceleration there must be further forces to balance weight.

These forces are; resolving parallel to the direction of travel - drag

Perpendicular to the direction of travel - lift
[Notice lift ≠ up!].

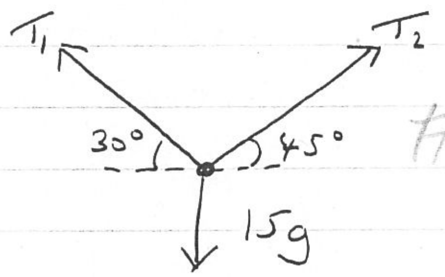
Example 5.2

A mass of 15 kg is suspended as shown, in equilibrium. Find the tension in each string



#34

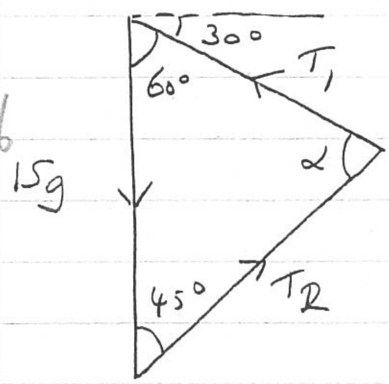
Force diagram on particle



#35

Method 1 triangle of forces: In equilibrium so the triangle must close.

#36



$\alpha = 180° - (60° + 45°)$
$= 75°$

By the sine rule $\frac{T_1}{\sin 45^\circ} = \frac{15g}{\sin 75^\circ}$

Hence $T_1 = 11g$.

and $\frac{T_2}{\sin 60} = \frac{15g}{\sin 75^\circ}$

Hence $T_2 = 13.4\,g$.

Method 2 Resolving forces (up & right +ve)

Vertically $-15g + T_1 \cdot \sin 30 + T_2 \cdot \sin 45 = 0$
Horizontally $-T_1 \cdot \cos 30 + T_2 \cos 45^\circ = 0$

So
$$\tfrac{1}{2} T_1 + \frac{\sqrt{2}}{2} T_2 = 15g$$
$$\frac{\sqrt{3}}{2} T_1 = \frac{\sqrt{2}}{2} T_2$$

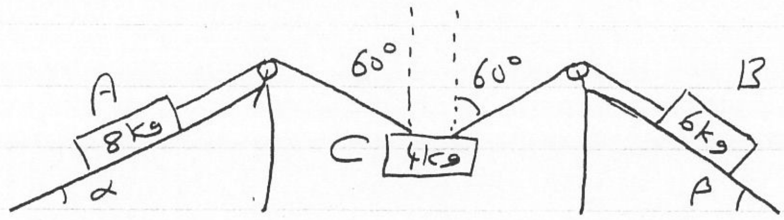
Sub for $\frac{\sqrt{2}}{2} T_2$,
$$T_1 \left(\tfrac{1}{2} + \frac{\sqrt{3}}{2} \right) = 15g$$
$$T_1 = \frac{30}{1+\sqrt{3}} g \approx 11g$$

$$T_2 = \sqrt{\frac{3}{2}}\, T_1 = 13.4\,g.$$

Triangle of forces	:	"Simple" (in simple cases)
Resolving forces	:	more general

Example

The diagram shows masses of 8kg and 6kg lying on smooth planes, at angles α and β respectively

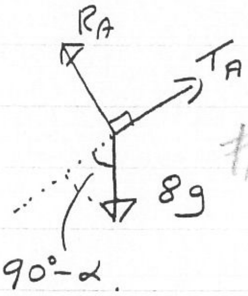


#36

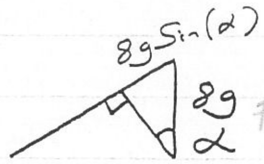
Light inextensible strings attach these masses over smooth pullies to a 4kg mass C. Both strings make an angle 60° to the vertical.

If the system is at rest then find α & β.

Forces on A:



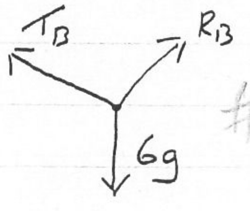
#37



#38

Parallel to the slope we have $T_A = 8g\sin(\alpha)$

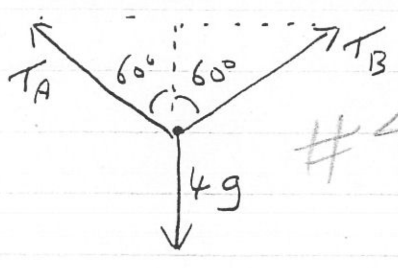
Forces on B:



#39

Parallel to the Slope $T_B = 6g\sin(\beta)$

Forces on C:



#40

Resolving in a vertical direction: $4g = (T_B + T_A)\cos(60^\circ)$

or $8g = T_A + T_B$.

Resolving in a horizontal direction $T_A \cdot \sin 60^\circ = T_B \sin 60^\circ$

or $T_A = T_B$

Let $T = T_A = T_B$.

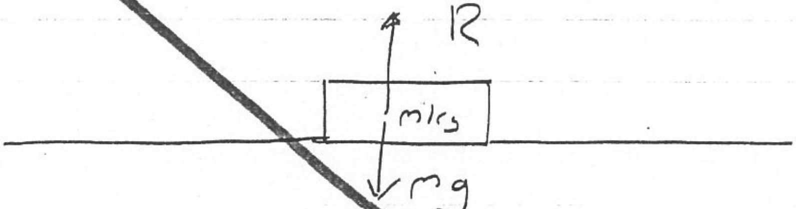
$$\left.\begin{aligned} T &= 8g \sin(\alpha) \\ T &= 6g \sin(\beta) \\ T &= 4g \end{aligned}\right] \text{ System to solve.}$$

$\sin(\alpha) = 1/2 \quad \Rightarrow \quad \alpha = 30^\circ$

$\sin(\beta) = 2/3 \quad \Rightarrow \quad \beta = 41^\circ$.

6 Friction.

Imagine a block sitting on a table in equilibrium.



Resolving vertically, $R = mg$.

Imagine a block slides along the table with initial velocity V. What happens? It soon accelerates to rest. So there must be a force which causes this accelation (N1)

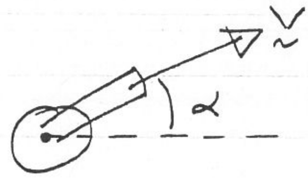
This force is called friction

Projectile motion

All/any vector can be resolved into different directions not just forces. E.g velocities.

Example

Imagine a projectile, P, being fired from a gun with muzzle velocity $\underset{\sim}{v}$ (at an angle α) to the horizontal. What happens next?



#41

What are the forces on P?

Air resistance mg (weight) #42

[Air resistance is (i) difficult to model ($R \propto |\underset{\sim}{v}|^2$)
(ii) Sometimes small (aerodynamic small fast P).
So we will ignore it!].

This means we are left with weight. P mg. #43

The only acceleration (change in velocity) is in the vertical direction. $\underset{\sim}{a} = -9.8\ ms^{-2}$ is constant.

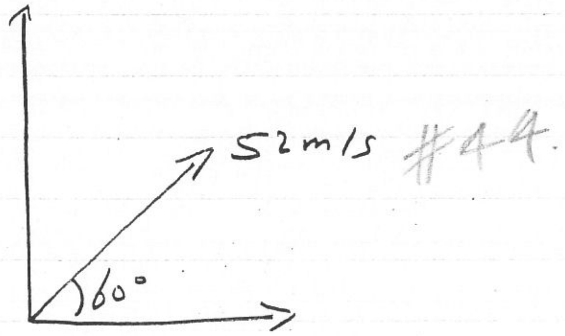
There is no change of horizontal velocity.

Example

A particle is projected from a point on a horizontal plane with an initial velocity of 52 m/s at an angle of 60°.

Find (i) greatest height reached
(ii) time to reach this height.

Method. Find components of velocity in the horizontal and vertical directions : resolve vectors



horizontal velocity: $52 \cdot \cos 60° = 26$ m/s
vertical velocity: $52 \cdot \sin 60° = 45$ m/s.

Now, for constant ~~velocity~~ acceleration of $-9.8\,ms^{-2}$

$$s = ut + \tfrac{1}{2}at^2$$

$$= 45 \cdot t - \frac{9.8}{2} t^2$$

$$= 45t - 4.9t^2$$

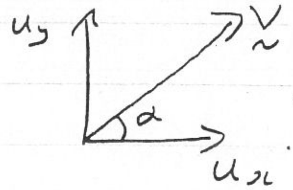
Find maximum of s. (i.e greatest height)
when $\frac{ds}{dt} = 0$ i.e when $V = 0$.

$$\frac{ds}{dt} = 45 - 9.8t = 0$$
$$t = 4.6\,s.$$

at which $s = 45 \times 4.6 - 4.9 \cdot (4.6)^2 = \underline{103m.}$

Path of a projectile

Initial speed u_x horizontally
u_y vertically



#45.

At time t.

$$x = S_x = u_x . t$$
$$y = S_y = u_y . t - \frac{g}{2} t^2$$

Parametric equation in t.

[GeoGebra].

Can we eliminate 't' to find the equation ?

$$x = u_x t$$

so $y = \frac{u_y}{u_x} . x - \frac{g}{2u_x^2} x^2$ a parabola.

[Many similar problems can be solved by resolving velocity]

Notice (i) $\tan(\alpha) = \frac{u_y}{u_x}$

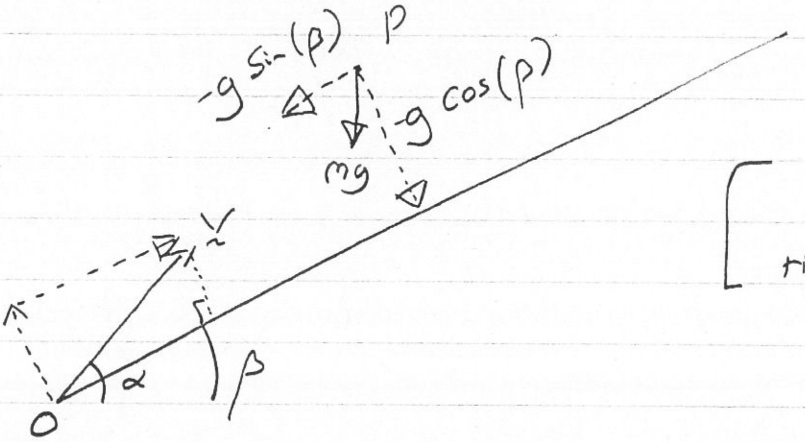
$$u_x = |\underset{\sim}{V}| . \cos(\alpha)$$

and (ii) $\frac{1}{\cos(\alpha)} = \sec^2\alpha = (1 + \tan^2\alpha)$

so $y = \tan(\alpha)\, x - \frac{g x^2 (1 + \tan^2\alpha)}{2|\underset{\sim}{V}|^2}$.

Projectiles on an inclined plane

#46



[Up +ve
right +ve]

A projectile is projected from a point O on an inclined plane, up the line of greatest slope.

Initial velocity is $\underset{\sim}{V}$, at angle α.
Plane angled at β.

Resolve parallel and perpendicular to the slope. !

	Initial Velocity	Forces
Parallel	$\lvert\underset{\sim}{V}\rvert\cos(\alpha-\beta)$	$-mg\sin(\beta)$
Perpendicular	$\lvert\underset{\sim}{V}\rvert\sin(\alpha-\beta)$	$-mg\cos(\beta)$

Why? We most often want time of contact with the plane, or distance from O along the plane.

Time of contact is when perpendicular distance $= 0$.

[Choose your coordinates with care!]