QUESTION 1. Consider the parabolic cylinder function, defined by

$$D_{-2m}(x) = \frac{1}{(m-1)!} x e^{-x^2/4} \int_0^\infty e^{-s} s^{m-1} (x^2+2s)^{-m-1/2} ds, \qquad (1)$$

where m is some positive integer.
Let $s = x^2 S$ and hence $ds = x^2 dS$, we then have

$$\begin{aligned} D_{-2m}(x) &= \frac{1}{(m-1)!} x e^{-x^2/4} \int_0^\infty e^{-x^2 S} x^{2(m-1)} S^{m-1} x^{-2m-1} (1+2S)^{-m-\frac{1}{2}} dS, \\ &= \frac{1}{(m-1)!} e^{-x^2/4} \int_0^\infty \frac{S^{m-1}}{(1+2S)^{m+\frac{1}{2}}} e^{-x^2 S} dS. \qquad (2) \end{aligned}$$

Since we have $x \gg 1$ we can now use Laplace's method, which shows that the integral in (2) is dominated by a contribution from the nieghbourhood of $S = 0$ and so

$$D_{-2m}(x) \sim \frac{1}{(m-1)!} e^{-x^2/4} \int_0^\infty S^{m-1} e^{-x^2 S} dS. \qquad (3)$$

In order to evaluate this integral we now let $s = x^2 S$ and $ds = x^2 dS$ and so

$$\begin{aligned} D_{-2m}(x) &\sim \frac{1}{(m-1)!} x e^{-x^2/4} \int_0^\infty x^{-2(m-1)} s^{m-1} e^{-s} x^{-2} ds, \\ &\sim \frac{1}{(m-1)!} e^{-x^2/4} x^{-2m} \int_0^\infty s^{m-1} e^{-s} ds, \\ &\sim \frac{\Gamma(m)}{(m-1)!} e^{-x^2/4} x^{-2m} \sim x^{-2m} e^{-x^2/4}. \qquad (4) \end{aligned}$$

QUESTION 2. The maximum of the exponent coccurs at $t = 0$. hence applying Laplace's method gives

$$I(x) \sim \int_0^\infty e^{-xt^n}(1 - t + \cdots)dt.$$

We then use the substitution $s = xt^n$ and so $nxt^{n-1}dt = ds$ whence we have

$$I(x) \sim \int_0^\infty \frac{e^{-s}}{nxt^{n-1}}ds = \frac{1}{nx^{1/n}}\int_0^\infty e^{-s}s^{1/n-1}ds = \frac{\Gamma(1/n)}{nx^{\frac{1}{n}}}.$$

JU 29/10/11